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Inverse Laplace Transforms of Osculatory and Hyperosculatory Interpolation Polynomials

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In the numerical calculation of $f(t)$, the inverse Laplace transform of $F(p)$, where $f(t) = (1/2\pi i)\int_{c-i\infty}^{c+i\infty} e^{pt} F(p)\, dp$, sufficient accuracy is usually obtainable when $p^s F(p)$, $s > 0$, is replaced by an interpolating polynomial in $1/p$. From the values of $F(p)$ with $F'(p)$, or with $F'(p)$ and $F''(p)$, for p at points equally spaced on the real axis, an osculatory or hyperosculatory interpolation polynomial for $p^s F(p)$, namely $L_{2n-1}(x)$ or $L_{3n-1}(x)$, where $x = 1/p$, is obtained in barycentric form. Then $f(t)$ is calculated by a Gaussian-type quadrature formula employing complex values of L_{2n-1} or L_{3n-1}, instead of $p^sF(p)$, which may be unknown or more difficult to compute. For calculating L_{2n-1} and L_{3n-1}, auxiliary coefficients, suitable for economical storage in the program, are given exactly for $n = 2(1)11$ and $n = 2(1)7$, furnishing up to 21st and 20th degree accuracy, respectively.

INTRODUCTION

For a given function $F(p)$, its inverse Laplace transform $f(t)$ is expressible as

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}F(p)\, dp, \tag{1}$$

where c is a real constant that is greater than the real part of each of the singular points of $F(p)$. Usually, $c > 0$, but we may have also $c \leqslant 0$, so long as for $f(t)$ satisfying Dirichlet's conditions in any finite positive interval, the integral $\int_0^\infty e^{-ct} f(t)\, dt$ is absolutely convergent [1, p. 75]. For a thorough survey of the many methods for the numerical evaluation of $f(t)$ with 183 references, see [2]. For an exhaustive and more up-to-date bibliography with 176 items, see [3]. This article is concerned with a numerical method that utilizes $F(p)$ and $F'(p)$ (osculatory case) or $F(p)$, $F'(p)$, and $F''(p)$ (hyperosculatory case) at equally spaced points on the real axis in conjunction with a Gaussian-type quadrature formula. An earlier work [4] was based upon the numerical values of $F(p)$ alone, at the points $p = 1, 2,..., n, \infty$, assuming that in (1) we may approximate $F(p)$ by $P_n(1/p)$, an $(n + 1)$-point nth degree Lagrangian interpolation polynomial in $1/p$. The condition $F(\infty) = 0$ was satisfied by the absence of a constant term in $P_n(1/p)$. A computer-adapted version of [4] was given in [5]. Piessens [6] treats a somewhat more general approximation than that in [4] by assuming that in (1), $F(p) = p^{-s} \times$ an interpolation polynomial in $1/p$ for various values of $s > 0$. Many numerical tests confirmed the practicality of the $P_n(1/p)$ approximation, even when it was compared with more accurate methods [4, 6, 7].

Whenever $F(p)$ may be approximated in (1) by polynomials in $1/p$, without a constant term, even if the degree must be quite high in order to obtain $f(t)$ accurately, we have the very efficient Gaussian-type quadrature

$$f(t) = (1/2\pi i t) \int_{c'-i\infty}^{c'+i\infty} e^p \phi(p)\, dp \sim (1/t) \sum_{i=1}^{n} A_i \phi(p_i), \tag{2}$$

where $\phi(p) = F(p/t)$, which is exact whenever $F(p)$, and consequently $\phi(p)$, are polynomials of the $(2n)$th degree in $1/p$ without a constant term [8, 9]. The p_i and A_i in (2) are all, with a single exception for each odd n, located in the complex plane. Tables of p_i and A_i, as well as $1/p_i$, are given to single precision in [8] and double precision in [9]. The most extensive tabulation, by Stroud, gives p_i and $A_i' \equiv A_i/p_i$ for $n = 2(1)\,24$, to $30S$ [10, pp. 307–315], the reason for A_i' instead of A_i being that Stroud sets $\phi(p) = (p/t)\,F(p/t)$ so that

$$f(t) = (1/2\pi i) \int_{c'-i\infty}^{c'+i\infty} (e^p/p)\, \phi(p)\, dp \sim \sum_{i=1}^{n} A_i' \phi(p_i), \tag{2$'$}$$

which is now exact whenever $\phi(p)$ is any polynomial of the $(2n - 1)$th degree in $1/p$. In the more general case when $F(p)$ is approximable by $p^{-s} \times$ a polynomial in $1/p$, $s > 0$, setting $\phi(p) = (p/t)^s\, F(p/t)$, we have

$$f(t) = (1/2\pi i t^{1-s}) \int_{c'-i\infty}^{c'+i\infty} (e^p/p^s)\, \phi(p)\, dp \sim \sum_{i=1}^{n} A_i' \phi(p_i). \tag{2$''$}$$

For (2″), there are tables of p_i and A_i' for $s = 1(1)\,5$, $n = 1(1)\,15$, to $20S$ [11, pp. 49–62] and for $s = 0.01(0.01)\,3$, $s \neq 1, 2, 3$, for $n = 1(1)\,10$, to $8S$ for p_i and $7S$ for A_i' [11, pp. 63–262]; also, for $s = 0.1(0.1)\,3\ (0.5)\,4$ and 16 fractions $\leqslant 10/3$, $n = 6(1)\,12$), to $16S$ [12].

In many cases, (2), (2′), or (2″) may be inconvenient when $F(p)$, for p complex, is either unknown or difficult to calculate. Now, if $p^sF(p)$ is replaceable in (1) by a suitable interpolating function, based upon values of $F(p)$ that are known for p just on the real axis, calculating that interpolating function for some complex argument, say, t/p_i (cf. (4) and (13) below) may be easier than calculating $(p_i/t)^s\, F(p_i/t)$ in the complex plane.

Frequently, we are given or readily can calculate $F(p)$ with either $F'(p)$ alone, or both $F'(p)$ and $F''(p)$ on the real axis at the conveniently located and equally spaced integral points $p = j, j = 1(1)\,n$.[1] Some functions $F(p)$ occur naturally for integral values of p. Other functions $F(p)$ may be readily available from previously calculated tables whose arguments are at equal intervals. Then, when $F(p)$, with $F'(p)$ or with $F'(p)$ and $F''(p)$, satisfies simple difference equations, it is usually easier to generate $F(j)$ with $F'(j)$, or with $F'(j)$ and $F''(j)$, than to calculate $(p_i/t)^s F(p_i/t)$ for (2), (2′), or (2″). On the basis of the test examples in [4, 6, 7], where in (1) either $F(p)$ alone or $p^s F(p)$, $s > 0$, was replaced by an interpolation polynomial in $1/p$, say $L_n(1/p)$, based on just real values of p, we should expect much greater accuracy by replacing in (1) $p^s F(p)$,[2] $s > 0$, by $L_{2n-1}(1/p)(L_{3n-1}(1/p))$, a $(2n-1)$th $((3n-1)$th) degree osculatory (hyperosculatory) interpolation polynomial in the variable $1/p$, obtained from $F(j)$ and $F'(j)(F(j), F'(j)$ and $F''(j))$ for n integral values of j, where n is not too large.

It should be emphasized that once we admit the accuracy of the approximation $(1/2\pi i)\int_{c-i\infty}^{c+i\infty} e^{pt}F(p)\,dp \sim (1/2\pi i)\int_{c-i\infty}^{c+i\infty} e^{pt}p^{-s}L_{2n-1}(1/p)\,dp$, or $(1/2\pi i)\int_{c-i\infty}^{c+i\infty} e^{pt}p^{-s} \times L_{3n-1}(1/p)\,dp$, where $L_{2n-1}(1/p)$ or $L_{3n-1}(1/p)$ is real, having been determined by osculatory or hyperosculatory interpolation on the real axis, in the Gaussian-type quadrature applied to the second or third integral (cf. (4) or (13) below), which is exact for polynomials, the complex points t/p_i may be very far from the points of interpolation on the real axis, and $L_{2n-1}(t/p_i)$ or $L_{3n-1}(t/p_i)$ may differ very much from $(p_i/t)^s F(p_i/t)$.[3] The only caution to be observed is with the possible loss of significant figures in the course of the computation when t/p_i, the arguments in the interpolation polynomials in (4) or (13), are far from the interpolation points $1/j$. But, for $s = 1$ (implicit in (2), explicit in (2′)), which is by far the most important case, the tables in [9] and surely those in [10] have enough significant figures for almost any conceivable example. For nonintegral s, the single-precision tabulation of p_i and A_i in [11], in certain cases might not provide enough significant figures.

The interpolation points in [4] are the integers $j = 1(1)\,n$ and $j = \infty$. Now, for osculatory and hyperosculatory interpolation, the assumption that $F(p) \sim p^{-s}P_n(1/p)$, $s > 0$, would immediately imply, besides $F(\infty) = 0$, also, $F^{(k)}(\infty) = 0$, $k \geqslant 1$. Such information is useless because it cannot yield any knowledge about derivatives of $L_{2n-1}(1/p)$ or $L_{3n-1}(1/p)$ with respect to $1/p$, for $1/p = 0$. Therefore, here, we drop the interpolation point $j = \infty$ and base

[1] Given $F(jh)$ with $F'(jh)$, or $F(jh)$ with both $F'(jh)$ and $F''(jh)$, instead of $F(j)$, $F'(j)$, and $F''(j)$, we change the variables in (1) to $p' = p/h$ and $t' = th$. Then, if $G(p') = F(p) = F(hp')$, we have $G(j) = F(jh)$, $G'(j) = hF'(jh)$, $G''(j) = h^2F''(jh)$, and $f(t) = hg(t') = (h/2\pi i)\int_{c''-i\infty}^{c''+i\infty} e^{pt'}\,G(p)\,dp$.

[2] "$F(p)$ alone" requires the condition of no constant term in the interpolating polynomial in $1/p$, which is more convenient to avoid in the osculatory and hyperosculatory cases (cf. paragraph after next).

[3] The variable in the osculatory and hyperosculatory interpolation polynomials for $p^sF(p)$ is $x = 1/p$, the interpolation points $x_j = 1/j$ chosen *before* we replace e^{pt} by e^p and $p^sF(p) \sim L_{2n-1}(1/p)$ or $L_{3n-1}(1/p)$ by $L_{2n-1}(t/p)$ or $L_{3n-1}(t/p)$.

our approximation upon $F(j)$ and $F'(j)$, or $F(j)$, $F'(j)$, and $F''(j)$, for $j = 1(1)n$. Considering $F(p) = p^{-s}\{p^sF(p)\}$, $s > 0$, we have the weight function $e^p p^{-s}$ multiplying $p^sF(p)$, which is either exactly or closely approximable by a polynomial in the variable $x = 1/p$. The osculatory and hyperosculatory interpolating polynomials in x are determined by $(d/dx)\{p^sF(p)\}\,|_{x=1/j}$ and $(d^2/dx^2)\{p^sF(p)\}\,|_{x=1/j}$ along with $j^sF(j)$, $j = 1(1)n$. They are expressible in terms of $F(j)$, $F'(j)$, and $F''(j)$ as follows:

$$L_{2n-1}(1/j) \text{ or } L_{3n-1}(1/j) = j^sF(j), \tag{3}$$

$$L'_{2n-1}(1/j) \text{ or } L'_{3n-1}(1/j) = (d/dx)\{p^sF(p)\}\,|_{x=1/j} = -sj^{s+1}F(j) - j^{s+2}F'(j), \tag{3'}$$

and

$$\begin{aligned} L''_{3n-1}(1/j) &= (d^2/dx^2)\{p^sF(p)\}\,|_{x=1/j} \\ &= s(s+1)\,j^{s+2}F(j) + (2s+2)\,j^{s+3}F'(j) + j^{s+4}F''(j). \end{aligned} \tag{3''}$$

In the most widely used case, $s = 1$, the right members of (3′) and (3″) are $-j^2F(j) - j^3F'(j)$ and $2j^3F(j) + 4j^4F'(j) + j^5F''(j)$, respectively.

The purpose of this article is to give a convenient computer-adapted method of calculating inverse Laplace transforms by replacing $p^sF(p)$ by the barycentric form of an osculatory or a hyperosculatory interpolation polynomial in the variable $x = 1/p$ and by employing the Gaussian quadrature formulas that are tabulated in [8–11].[4] To facilitate the computation of the barycentric forms, auxiliary coefficients, which may be stored economically in the program, have been calculated exactly to furnish up to 21st or 20th degree accuracy. It is also worth noting that the interpolation formulas, which are given here in conjunction with the calculation of inverse Laplace transforms, have many other applications involving reciprocal arguments.

Osculatory Interpolation

In addition to the cases where $F(j)$ and $F'(j)$, $j = 1(1)\,n$ are specified initially, often, from $F(j)$ alone, we may readily obtain $F'(j)$ when $F(p)$ satisfies a simple first-order differential equation.

[4] For $s = 1$, because the interpolating polynomials in $1/p$ that are given here will generally have a constant term, we require extra divisions by p_i in order to use the Christoffel numbers A_i in [8, 9]. This does not occur in (2′) or (2″), which occur in [10] or [11]. Henceforth, the unprimed A_i will denote the Stroud–Krylov Christoffel numbers.

In (1), we replace $p^s F(p)$ by the osculatory interpolation polynomial $L_{2n-1}(x)$, $x = 1/p$, where $L_{2n-1}(1/j)$ and $L'_{2n-1}(1/j)$, $j = 1(1)\,n$, are given by (3) and (3′). Then, we find

$$f(t) = (1/2\pi i)\int_{c-i\infty}^{c+i\infty} (e^{pt}/p^s)\{p^s F(p)\}\,dp \sim (1/2\pi i)\int_{c-i\infty}^{c+i\infty} (e^{pt}/p^s)\, L_{2n-1}(1/p)\,dp$$

$$= (1/2\pi i t^{1-s})\int_{c'-i\infty}^{c'+i\infty} (e^{p}/p^s)\, L_{2n-1}(t/p)\,dp$$

$$= t^{s-1}\sum_{i=1}^{n} A_i L_{2n-1}(t/p_i). \tag{4}$$

We calculate $L_{2n-1}(t/p_i)$ efficiently from the barycentric form of Hermite's n-point osculatory interpolation formula. Since it is exact for any $(2n-1)$th degree polynomial, we have, for any x_j, $j = 1(1)\,n$, and any x,

$$L_{2n-1}(x) = \sum_{j=1}^{n} \{L_j^{(n)}(x)\}^2\{[1 - 2L_j^{(n)\prime}(x_j)(x - x_j)]\, L_{2n-1}(x_j) + (x - x_j)\, L'_{2n-1}(x_j)\} \tag{5}$$

where

$$L_j^{(n)}(x) = \prod_{k=1,k\neq j}^{n} (x - x_k)\Big/\prod_{k=1,k\neq j}^{n} (x_j - x_k). \tag{6}$$

The barycentric form of (5) is

$$L_{2n-1}(x) = \sum_{j=1}^{n} [\alpha_j L_{2n-1}(x_j) + \beta_j L'_{2n-1}(x_j)]\Big/\sum_{j=1}^{n} \alpha_j, \tag{7}$$

where

$$\alpha_j = d_j/(x - x_j)^2 - 2L_j^{(n)\prime}(x_j)\, d_j/(x - x_j), \qquad j = 1(1)\,n, \tag{8}$$

$$\beta_j = d_j/(x - x_j), \qquad j = 1(1)\,n, \tag{9}$$

and

$$d_j = \left\{1\Big/\prod_{k=1,k\neq j}^{n} (x_j - x_k)\right\}^2, \qquad j = 1(1)\,n. \tag{10}$$

For $x_j = 1/j$, for each n, d_j and $-2L_j^{(n)\prime}(x_j)\, d_j$, $j = 1(1)\,n$, are all multiplied by a rational number $r(n)$ to obtain (α_j and β_j now denoting $r(n)\,\alpha_j$ and $r(n)\,\beta_j$)

$$\alpha_j = a_j/(x - 1/j)^2 + b_j/(x - 1/j), \qquad j = 1(1)\,n, \tag{11}$$

and

$$\beta_j = a_j/(x - 1/j), \qquad j = 1(1)\,n, \tag{12}$$

where now, a_j and b_j are integers whose g.c.d. $= 1$ for every n. The values of $r(n)$ are given in the following schedule:

n	2	3	4	5	6	7	8	9	10	11
$r(n)$	1/4	1/9	3/16	6/25	5/6	10/49	35/32	140/81	63/5	1260/121.

Tables Ia–Ij give the exact values of a_j and b_j, $j = 1(1)\,n$, for $n = 2(1)\,11$, furnishing up to 21th degree accuracy in $L_{2n-1}(x)$. Thus, $L_{2n-1}(t/p_i)$ in (4) is found by first setting $x = t/p_i$ in (11) and (12), where for each $i = 1(1)\,n$ we have $j = 1(1)\,n$, and then employing (7), where $L_{2n-1}(x_j) = L_{2n-1}(1/j)$ and $L'_{2n-1}(x_j) = L'_{2n-1}(1/j)$, $j = 1(1)\,n$, are given by (3) and (3′).

TABLE Ia

$n = 2$

j	a_j	b_j
1	1	−4
2	1	4

TABLE Ib

$n = 3$

j	a_j	b_j
1	1	−7
2	16	−128
3	9	135

TABLE Ic

$n = 4$

j	a_j	b_j
1	3	−29
2	432	−6912
3	2187	−19683
4	768	26624

TABLE Id

$n = 5$

j	a_j	b_j
1	6	−73
2	6144	−1 39264
3	1 57464	−37 79136
4	3 93216	−20 97152
5	93750	60 15625

TABLE Ie

$n = 6$

j	a_j	b_j
1	30	−437
2	1 92000	−55 04000
3	196 83000	−7085 88000
4	1966 08000	−57671 68000
5	2929 68750	12207 03125
6	503 88480	52605 57312

TABLE If

$n = 7$

j	a_j	b_j
1	10	−169
2	3 68640	−126 32064
3	1328 60250	−61780 01625
4	41943 04000	−20 13265 92000
5	2 19726 56250	−67 74902 34375
6	2 17678 23360	44 40635 96544
7	28247 52490	44 09438 63689

TABLE Ig

$n = 8$

j	a_j	b_j
1	70	−1343
2	140 49280	−5563 51488
3	1 64055 83670	−92 03532 43887
4	143 86462 72000	−9207 33614 08000
5	2093 50585 93750	−1 20376 58691 40625
6	6719 72707 12320	−1 85464 46716 60032
7	4747 56150 99430	2 09367 46258 84863
8	481 03633 71520	1 05773 01859 20512

TABLE Ih

$n = 9$

j	a_j	b_j
1	140	−3001
2	1468 00640	−65682 80064
3	52 49786 77440	−3417 61119 01344
4	11785 39026 02240	−9 23974 59640 15616
5	4 18701 17187 50000	−334 96093 75000 00000
6	34 40500 26047 07840	−2188 15816 56594 18624
7	74 44176 44759 06240	−1406 94934 85946 27936
8	39 40649 67394 91840	2990 39015 25740 09344
9	3 20275 09436 94540	948 95222 78242 37241

TABLE Ii

$n = 10$

j	a_j	b_j
1	1260	−29809
2	66886 04160	−33 27102 81216
3	7029 35735 24160	−5 17862 79809 15616
4	38 18466 44431 25760	−3502 80655 15827 36384
5	3052 33154 29687 50000	−3 05233 15429 68750 00000
6	56432 80552 23720 34560	−52 82110 59689 40224 34816
7	2 95459 36320 48718 66560	−193 72285 58079 94320 50784
8	4 59637 37796 94328 21760	−18 91079 49793 13807 52384
9	1 89119 24047 42188 92460	219 93216 81543 39842 01009
10	12600 00000 00000 00000	48 61000 00000 00000 00000

TABLE Ij

$n = 11$

j	a_j	b_j
1	1260	−32581
2	3 30301 44000	−180 44944 38400
3	98850 33776 83500	−80 97960 88475 14725
4	1246 84618 58979 84000	−1 30051 99455 19497 21600
5	2 11967 46826 17187 50000	−250 82817 07763 67187 50000
6	81 26323 99522 15729 76640	−9751 58879 42658 87571 96800
7	904 84429 98149 20091 34000	−94164 13013 40726 84172 11600
8	3268 53246 55604 11176 96000	−2 05201 58107 59450 52176 38400
9	3829 66461 96029 32572 31500	66225 84317 18478 55411 24725
10	1260 00000 00000 00000 00000	2 08900 00000 00000 00000 00000
11	70 05495 81500 02116 66060	34243 58633 29717 88613 79381

Hyperosculatory Interpolation

Besides the situations where $F(j)$, $F'(j)$, and $F''(j)$, $j = 1(1)\,n$, are given initially or where $F'(j)$ and $F''(j)$ are obtainable from $F(j)$ when $F(p)$ satisfies a first-order differential equation, there also may be instances when we have $F(j)$ and $F'(j)$ initially and $F''(j)$ is obtainable from them when $F(p)$ satisfies a simple second-order ordinary differential equation.

In (1), we replace $p^sF(p)$ by the hyperosculatory interpolation polynomia $L_{3n-1}(x)$, $x = 1/p$, where $L_{3n-1}(1/j)$, $L'_{3n-1}(1/j)$, and $L''_{3n-1}(1/j)$, $j = 1(1)\,n$, are givei by (3), (3′), and (3″). Then, we find

$$f(t) = (1/2\pi i)\int_{c-i\infty}^{c+i\infty} (e^{pt}/p^s)\{p^sF(p)\}\,dp \sim (1/2\pi i)\int_{c-i\infty}^{c+i\infty} (e^{pt}/p^s)\,L_{3n-1}(1/p)\,dp$$

$$= (1/2\pi i t^{1-s})\int_{c'-i\infty}^{c'+i\infty} (e^{p}/p^s)\,L_{3n-1}(t/p)\,dp$$

$$= t^{s-1}\sum_{i=1}^{[(3n+1)/2]} A_i L_{3n-1}(t/p_i), \tag{13}$$

where $[(3n+1)/2]$, the nearest integer to $(3n+1)/2$, is the smallest number of points in the Gaussian quadrature formula that will provide at least $(3n-1)$th degree accuracy. Hermite's n-point hyperosculatory interpolation formula, exact for any $(3n-1)$th degree polynomial, when applied to $L_{3n-1}(x)$ itself, gives for any x_j, $j = 1(1)\,n$, and any x,

$$L_{3n-1}(x) = \sum_{j=1}^{n} \{L_j^{(n)}(x)\}^3\{[1 - 3L_j^{(n)\prime}(x_j)(x - x_j) + (6L_j^{(n)\prime}(x_j)^2 - \tfrac{3}{2}L_j^{(n)\prime\prime}(x_j))(x - x_j)^2]\,L_{3n-1}(x_j) + [(x - x_j) - 3L_j^{(n)\prime}(x_j)(x - x_j)^2]\,L'_{3n-1}(x_j) + \tfrac{1}{2}(x - x_j)^2\,L''_{3n-1}(x_j)\}, \tag{14}$$

where $L_j^{(n)}(x)$ is given by (6). The barycentric form of (14) is

$$L_{3n-1}(x) = \sum_{j=1}^{n} [\alpha_j L_{3n-1}(x_j) + \beta_j L'_{3n-1}(x_j) + \tfrac{1}{2}\gamma_j L''_{3n-1}(x_j)]\Big/\sum_{j=1}^{n}\alpha_j, \tag{15}$$

where

$$\alpha_j = d_j/(x - x_j)^3 - 3L_j^{(n)\prime}(x_j)\,d_j/(x - x_j)^2 + [6L_j^{(n)\prime}(x_j)^2 - \tfrac{3}{2}L_j^{(n)\prime\prime}(x_j)]\,d_j/(x - x_j), \quad j = 1(1)\,n, \tag{16}$$

$$\beta_j = d_j/(x - x_j)^2 - 3L_j^{(n)\prime}(x_j)\,d_j/(x - x_j), \qquad j = 1(1)\,n, \tag{17}$$

$$\gamma_j = d_j/(x - x_j), \qquad j = 1(1)\,n, \tag{18}$$

and

$$d_j = \left\{1\Big/\prod_{k=1,k\neq j}^{n} (x_j - x_k)\right\}^3, \qquad j = 1(1)\,n. \tag{19}$$

For $x_j = 1/j$, for each n, d_j, $-3L_j^{(n)\prime}(x_j)\, d_j$, and $[6L_j^{(n)\prime}(x_j)^2 - \frac{3}{2}L_j^{(n)\prime\prime}(x_j)]\, d_j$, $j = 1(1)\, n$, are all multiplied by a rational number $r(n)$ to obtain (α_j, β_j, and γ_j now denoting $r(n)\, \alpha_j$, $r(n)\, \beta_j$, and $r(n)\, \gamma_j$)

$$\alpha_j = a_j/(x - 1/j)^3 + b_j/(x - 1/j)^2 + c_j/(x - 1/j), \qquad j = 1(1)\, n, \tag{20}$$

$$\beta_j = a_j/(x - 1/j)^2 + b_j/(x - 1/j), \qquad j = 1(1)\, n, \tag{21}$$

and

$$\gamma_j = a_j/(x - 1/j), \qquad j = 1(1)\, n, \tag{22}$$

where now, a_j, b_j, and c_j are integers whose g.c.d. = 1 for every n. The values of $r(n)$ are given in the following schedule:

n	2	3	4	5	6	7
$r(n)$	1/8	2/27	3/32	12/125	25/18	600/343.

Tables IIa–IIf gives the exact values of a_j, b_j, and c_j, $j = 1(1)\, n$, for $n = 2(1)\, 7$, furnishing up to 20th degree accuracy in $L_{3n-1}(x)$. To obtain $L_{3n-1}(t/p_i)$ in (13), set $x = t/p_i$ in (20)–(22), where for each $i = 1(1)[(3n + 1)/2]$ we have $j = 1(1)\, n$, and then employ (15), where $L_{3n-1}(x_j) = L_{3n-1}(1/j)$, $L'_{3n-1}(x_j) = L'_{3n-1}(1/j)$, and $L''_{3n-1}(x_j) = L''_{3n-1}(1/j)$, $j = 1(1)\, n$, are given by (3), (3′), and (3″).

TABLE IIa

$n = 2$

j	a_j	b_j	c_j
1	1	−6	24
2	−1	−6	−24

TABLE IIb

$n = 3$

j	a_j	b_j	c_j
1	2	−21	129
2	−128	1536	−16896
3	54	1215	16767

TABLE IIc

$n = 4$

j	a_j	b_j	c_j
1	6	−87	703
2	−10368	2 48832	−38 56896
3	1 18098	−15 94323	430 46721
4	−24576	−12 77952	−391 90528

TABLE IId

$n = 5$

j	a_j	b_j	c_j
1	12	−219	2171
2	−3 93216	133 69344	−2668 62592
3	510 18336	−18366 60096	5 13116 91432
4	−2013 26592	16106 12736	−17 60936 59136
5	234 37500	22558 59375	12 50488 28125

TABLE IIe

$n = 6$

j	a_j	b_j	c_j
1	300	−6555	76577
2	−1536 00000	66048 00000	−15 95392 00000
3	15 94323 00000	−860 93442 00000	29809 85429 25000
4	−503 31648 00000	22145 92512 00000	−10 20054 73280 00000
5	915 52734 37500	5722 04589 84375	18 97811 88964 84375
6	−65 30347 00800	−10226 52341 45280	−9 07551 05722 85952

TABLE IIf

$n = 7$

j	a_j	b_j	c_j
1	600	−15210	2 03939
2	−42467 32800	21 82820 65920	−614 46259 99872
3	2905 65366 75000	−2 02669 34330 81250	83 84626 86427 96875
4	−5 15396 07552 00000	371 08517 43744 00000	−19488 84360 23296 00000
5	61 79809 57031 25000	−2858 16192 62695 31250	2 21378 80325 31738 28125
6	−60 93597 40010 49600	−1864 64080 44321 17760	−2 89436 12674 91454 32064
7	2 84853 69059 65800	666 98491 65318 92070	87462 32697 42842 02997

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